THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 4

1. Without using any software, sketch the graph of the following functions.

   (a) $f(x, y) = x^2 + y^2$

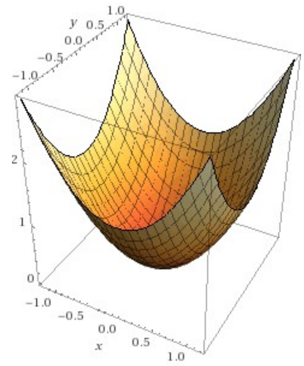
   (b) $f(x, y) = x^2 - y^2$

   (c) $f(x, y) = -x^2 - y^2$

   For each of the above function, determine whether $(0, 0)$ is a maximum or minimum point.

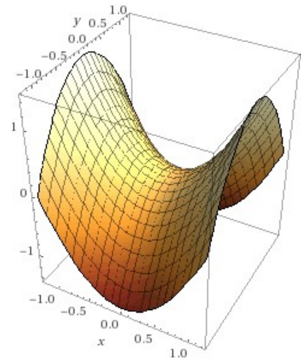
   Ans:

   (a) $f(x, y) = x^2 + y^2$



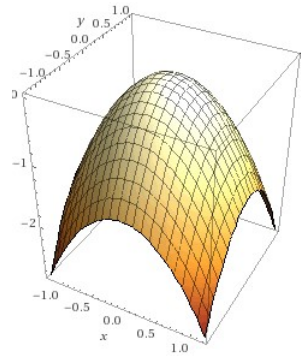
   $(0, 0)$ is a minimum point.

   (b) $f(x, y) = x^2 - y^2$



   $(0, 0)$ is a saddle point.

   (c) $f(x, y) = -x^2 - y^2$



   $(0, 0)$ is a maximum point.

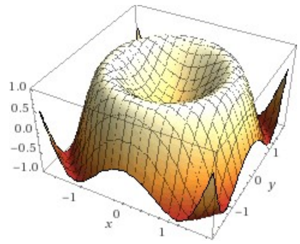
2. Let $f(x,y) = \sin(x^2+y^2)$.

   (a) Plot the graph of the function $f(x,y)$.

   (b) Describe the level set $L_{-1}(f)$, $L_0(f)$ and $L_1(f)$.

   Ans:

   (a) $f(x,y) = \sin(x^2+y^2)$



   (b) $L_{-1}(f) = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 = 2k\pi + 3\pi/2,\ k = 0,1,2,\ldots\}$
   $L_0(f) = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 = k\pi,\ k = 0,1,2,\ldots\}$
   $L_1(f) = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 = 2k\pi + \pi/2,\ k = 0,1,2,\ldots\}$

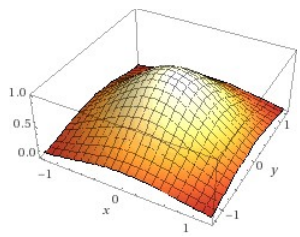
3. Let $f(x,y) = e^{-x^2-y^2}$.

   (a) Plot the graph of the function $f(x,y)$.

   (b) Describe the level set $L_c(f)$.

   Ans:

   (a) $f(x,y) = e^{-x^2-y^2}$



   (b) For $0 < c \leq 1$, we have $L_c(f) = \{(x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 = \ln(\frac{1}{c})\}$;
   otherwise, $L_c(f)$ is empty.

4. Let $f(x,y) = \begin{cases} 1 & \text{if } |x| = |y|; \\ 0 & \text{otherwise.} \end{cases}$

   (a) Sketch the graph of the function $f(x,y)$.

   (b) Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

   Ans:

   (a) The graph of $f(x,y)$ is almost the xy-plane, except that when (x,y) is a point lying on the two straight lines $x = y$ or $x = -y$, $f(x,y) = 1$. Therefore, you will see a cross lifting on.

   (b) Consider $\gamma_1(t) = (0,t)$ and $\gamma_2(t) = (t,t)$. Then, for $t \neq 0$, we have $\gamma_1(t) = 0$ and so $\lim_{t\to 0} f(\gamma_1(t)) = \lim_{t\to 0} 0 = 0$.
   On the other hand, $\lim_{t\to 0} f(\gamma_2(t)) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} 1 = 1$.
   We have $\lim_{t\to 0} f(\gamma_1(t)) \neq \lim_{t\to 0} f(\gamma_2(t))$ and so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

5. Let $f(x,y) = \dfrac{xy^2-1}{y-1}$. Prove that $\displaystyle\lim_{(x,y)\to(1,1)} f(x,y)$ does not exist.

Ans:

Consider $\gamma_1(t) = (1, 1+t)$ and $\gamma_2(t) = (1+t, 1+t)$. Then, we have

$$\lim_{t\to 0} f(\gamma_1(t)) = \lim_{t\to 0} \frac{2t+t^2}{t} = 2.$$

On the other hand,

$$\lim_{t\to 0} f(\gamma_2(t)) = \lim_{t\to 0} \frac{3t+3t^2+t^3}{t} = 3.$$

We have $\displaystyle\lim_{t\to 0} f(\gamma_1(t)) \neq \lim_{t\to 0} f(\gamma_2(t))$ and so $\displaystyle\lim_{(x,y)\to(1,1)} f(x,y)$ does not exist.

6. Determine whether each the following limit exists, if yes, find its value; if no, prove your assertion.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+x^2y+3xy^2+3y^3}{x^2+3y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$

Ans:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+x^2y+3xy^2+3y^3}{x^2+3y^2} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x^2+3y^2)}{x^2+3y^2} = \lim_{(x,y)\to(0,0)} x+y = 0$

(b) Let $f(x,y) = \dfrac{x^2+\sin^2 y}{2x^2+y^2}$.
Let $\gamma_1(t) = (t,0)$ and $\gamma_2(t) = (0,t)$, for $t\in\mathbb{R}$. Then, we have $\gamma_1(0) = \gamma_2(0) = (0,0)$.

$$\lim_{t\to 0}(f\circ\gamma_1)(t) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$
$$\lim_{t\to 0}(f\circ\gamma_2)(t) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0}\frac{\sin^2 t}{t^2} = \lim_{t\to 0}\left(\frac{\sin t}{t}\right)^2 = 1^2 = 1$$

$\displaystyle\lim_{t\to 0}(f\circ\gamma_1)(t) \neq \lim_{t\to 0}(f\circ\gamma_2)(t)$ and so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$ does not exist.

(c) Consider $(x,y)\in B_1^\circ(0,0)$, i.e. $0 < \sqrt{x^2+y^2} < 1$, we then have $x^2 \le x^2+y^2 < 1$. Therefore, for all $(x,y)\in B_1^\circ(0,0)$, we have

$$y^2 < x^2+y^2 < 1+y^2$$
$$\frac{1}{y^2+1} < \frac{1}{x^2+y^2} < \frac{1}{y^2}$$

Also, we have $-|y|^3 \le y^3 \le |y|^3$, so

$$-|y| = -\frac{|y|^3}{y^2} < \frac{y^3}{x^2+y^2} < \frac{|y|^3}{y^2} = |y|$$

Note that $\displaystyle\lim_{(x,y)\to(0,0)} -|y| = \lim_{(x,y)\to(0,0)} |y| = 0$.

By sandwich theorem, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$.

7. Determine whether each the following limit exists, if yes, find its value; if no, prove your assertion.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$;

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^4}$;

(d) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$.

Ans:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0$

(b) Let $r=\sqrt{x^2+y^2}$. Then,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} &= \lim_{r\to 0}\frac{\sin(r^2)}{r^2} \\ &= 1\end{aligned}$$

(c) Let $f(x,y)=\dfrac{x^2y}{x^3+y^4}$ and let $\gamma(t)=(t,mt)$, for $t\in\mathbb{R}$. Then,

$$\begin{aligned}\lim_{t\to 0} f(\gamma(t)) &= \lim_{t\to 0}\frac{mt^3}{t^3+m^4t^4} \\ &= \lim_{t\to 0}\frac{m}{1+m^4t} \\ &= m\end{aligned}$$

which depends on m.

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^4}$ does not exist.

(d) Let $f(x,y)=\dfrac{xy^2-1}{y-1}$ and let $\gamma(t)=(1+t,1+mt)$, for $t\in\mathbb{R}$. Then,

$$\begin{aligned}\lim_{t\to 0} f(\gamma(t)) &= \lim_{t\to 0}\frac{(1+t)(1+mt)^2-1}{(1+mt)-1} \\ &= \lim_{t\to 0}\frac{(2m+1)t+(m^2+2m)t^2+m^2t^3}{mt} \\ &= \lim_{t\to 0}\frac{(2m+1)+(m^2+2m)t+m^2t^2}{m} \\ &= \frac{2m+1}{m}\end{aligned}$$

which depends on m.

Therefore, $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$ does not exist.

8. Let $f(x,y)=\dfrac{xy^3}{x^3+y^5}$.

(a) i. Let $\gamma(t)=(t,mt)$, for $m\in\mathbb{R}$. Show that $\lim_{t\to 0} f(\gamma(t))=0$.

ii. Let $\gamma(t)=(0,t)$. Show that $\lim_{t\to 0} f(\gamma(t))=0$.

(b) Let $\gamma(t)=(t^3,t^2)$, for $m\in\mathbb{R}$. Show that $\lim_{t\to 0} f(\gamma(t))=1$.

Hence, determine whether $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not.

Ans:

(a) i. $\displaystyle\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0}\frac{m^3t^4}{t^3+m^5t^5} = \lim_{t\to 0}\frac{m^3t}{1+m^5t^2} = 0.$

ii. $\displaystyle\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0}\frac{(0)(t^3)}{0^3+t^5} = \lim_{t\to 0} 0 = 0.$

(b) $\displaystyle\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0}\frac{t^9}{t^9+t^{10}} = \lim_{t\to 0}\frac{1}{1+t} = 1.$

When (x, y) tends to $(0, 0)$ along any straight line, $f(x, y)$ tends to 0; but when (x, y) tends to $(0, 0)$ along the curve $\gamma(t) = (t^3, t^2)$, $f(x, y)$ tends to 1.

Therefore, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

9. (a) Prove that for all $u > 0$, we have

$$\frac{1}{1+u^2} < \frac{\tan^{-1} u}{u} < 1.$$

(b) Using the result in (a), evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(|x| + |y|)}{|x| + |y|}$.

Ans:

(a) Let $f(x) = \tan^{-1} x$ and let $u > 0$.

Note that f is continuous on $[0, u]$ and differentiable on $(0, u)$.

By mean value theorem, there exists $c \in [0, u]$ such that

$$\begin{aligned} \frac{f(u) - f(0)}{u - 0} &= f'(c) \\ \frac{\tan^{-1} u}{u} &= \frac{1}{1+c^2} \end{aligned}$$

Since $0 < c < u$, we have $1 < 1 + c^2 < 1 + u^2$ and $\dfrac{1}{1+u^2} < \dfrac{1}{1+c^2} < 1$. Therefore,

$$\frac{1}{1+u^2} < \frac{\tan^{-1} u}{u} < 1.$$

(b) If $(x, y) \neq (0, 0)$, then $|x| + |y| > 0$. By putting $u = |x| + |y|$ in the inequality obtained in (a), we have

$$\frac{1}{1 + (|x| + |y|)^2} < \frac{\tan^{-1}(|x| + |y|)}{|x| + |y|} < 1.$$

Note that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{1 + (|x| + |y|)^2} = \lim_{(x,y)\to(0,0)} 1 = 1.$

By sandwich theorem, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(|x| + |y|)}{|x| + |y|} = 1.$